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Open group transformations¹

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Abstract

Recent results on finite open group transformations are reviewed.

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Introduction

It is a pleasure for me to be at this conference dedicated to the memory of Efim Fradkin. I met Fradkin for the first time at Alushta, Crimea in April 1976 where he gave a talk on general constraints and their quantization. This memory is particularly pleasant to me now since the content of my talk is closely related to the highlights of Fradkin's general constraint theory. My talk is a review of some recent work I have done together with Igor Batalin, one of Fradkin's most frequent collaborators on these matters who unfortunately was unable to deliver a talk at this time due to family reasons. I will talk about finite open group transformations and then mainly as gauge transformations within the Batalin-Fradkin-Vilkovisky (BFV) framework. It is based on four papers [1]-[4] with emphasis on the second one which has the same title as this talk.

Let me first define what I mean by open groups. Consider a Hamiltonian formulation on a symplectic manifold Γ with coordinates z^A . Let $\theta_\alpha(z)$ be functions on Γ with Grassmann parities, $\varepsilon(\theta_\alpha) \equiv \varepsilon_\alpha = 0, 1$, satisfying the Poisson algebra

$$\{\theta_\alpha(z), \theta_\beta(z)\} = U_{\alpha\beta}^\gamma(z) \theta_\gamma(z), \quad (1)$$

where $U_{\alpha\beta}^\gamma(z)$ are restricted by the Jacobi identities. For $U_{\alpha\beta}^\gamma$ constants $\theta_\alpha(z)$ are generators of a Lie group while for $U_{\alpha\beta}^\gamma(z)$ functions $\theta_\alpha(z)$ are generators of open groups (sometimes called groupoids). The transformations generated by $\theta_\alpha(z)$ are symmetry transformations in general only if $\theta_\alpha(z) = 0$ at the level of the equations of motion. Open groups as symmetry groups are therefore in general gauge groups. Important open gauge groups appear in gravity, supergravity and p-branes. In this talk I will show that it is possible to integrate (1). I will explicitly construct an exponential representation of the group elements. This will be done directly at the quantum level. On our way we will find some remarkable mathematics.

The quantum Lie algebra - the BFV-BRST charge

The first question we have to answer is how (1) should be quantized in order to preserve the Lie algebra. The answer is given by the general BRST formulation given by Batalin, Fradkin and Vilkovisky – the BFV-formalism [5]. They require us first to ghost extend the manifold by generalized Faddeev-Popov ghosts \mathcal{C}^α and their canonical momenta \mathcal{P}_α with Grassmann parities $\varepsilon(\mathcal{C}^\alpha) = \varepsilon_\alpha + 1$, $\varepsilon_\alpha \equiv \varepsilon(\theta_\alpha)$. Then they require us to construct the odd BFV-BRST charge

$$\Omega = \mathcal{C}^\alpha \theta_\alpha(z) + \frac{1}{2} \mathcal{C}^\beta \mathcal{C}^\alpha U_{\alpha\beta}^\gamma \mathcal{P}_\gamma (-1)^{\varepsilon_\gamma + \varepsilon_\beta} + \dots, \quad (2)$$

where the dots indicates terms of higher powers in the ghost momenta. These terms are determined by the condition

$$\{\Omega, \Omega\} = 0, \quad (3)$$

which is equivalent to (1) together with all its Jacobi identities. That this always is possible is, I think, Fradkin's most important contribution to the constraint theory. The maximal power of \mathcal{P}_α in Ω defines the rank of the group which is a way to roughly classify open groups. For $\text{rank} \geq 2$ we have open groups. However, even rank one contains nontrivial open groups, the quasi groups [6]. Batalin and Fradkin have shown that for finite

dimensional manifolds with globally defined canonical coordinates it is always possible to quantize such that $\hat{\Omega}^2 = 0$ where $\hat{\Omega}$ is the corresponding operator defined up to \hbar -corrections [7]. (For infinite degrees of freedom we may have anomalies.) $\hat{\Omega}$ in \mathcal{CP} -ordered form is (“hat” is suppressed in the following)

$$\Omega = \sum_{i=0}^N \Omega'_i, \quad \Omega'_0 \equiv \mathcal{C}^\alpha \theta'_\alpha,$$

$$\Omega'_i \equiv \mathcal{C}^{\alpha_{i+1}} \dots \mathcal{C}^{\alpha_1} \Omega'^{\beta_1 \dots \beta_i}_{\alpha_1 \dots \alpha_{i+1}} \mathcal{P}_{\beta_1} \dots \mathcal{P}_{\beta_i}, \quad (4)$$

where $\theta'_\alpha = \theta_\alpha + \hbar$ -corrections. $\Omega^2 = 0$ requires then $[\theta'_\alpha, \theta'_\beta] = i\hbar U'^\gamma_{\alpha\beta} \theta'_\gamma$, where $U'^\gamma_{\alpha\beta} = 2(-1)^{\varepsilon_\beta + \varepsilon_\gamma} \Omega'^\gamma_{\alpha\beta} = U^\gamma_{\alpha\beta} + \hbar$ -corrections. Thus, the operator Ω indeed represents the quantum Lie algebra if $\Omega^2 = 0$.

A scenario for possible forms of group elements

I am now going to integrate the Lie algebra inherent in the BFV-BRST charge operator Ω . However, let me first give a scenario for finite open group transformations which is exactly true for Lie group theories before I embark on the actual derivation. Imagine we have a finite group element $U(\phi)$ where ϕ^α are group parameters, $\varepsilon(\phi^\alpha) = \varepsilon_\alpha \equiv \varepsilon(\theta_\alpha)$. It is natural to expect the existence of an exponential representation

$$U(\phi) = \exp \left\{ \frac{i}{\hbar} F(\phi) \right\}, \quad (5)$$

where $F(0) = 0$ by a choice of parametrization ($U(0) = \mathbf{1}$), and where $[\Omega, F(\phi)] = 0$ which is required by applications to gauge groups. We also assume $[G, F(\phi)] = 0$ where G is the ghost charge ($[G, \Omega] = i\hbar\Omega$) so that group transformations do not change ghost numbers. Since $U(\phi)$ performs gauge transformations in gauge theories we have

$$\langle ph | U(\phi) | ph \rangle = \langle ph | ph \rangle, \quad (6)$$

when $\Omega | ph \rangle = 0$. This requires

$$F(\phi) = \frac{1}{i\hbar} [\Omega, \rho(\phi)]. \quad (7)$$

Furthermore, since $\tilde{\theta}_\alpha \equiv (i\hbar)^{-1} [\Omega, \mathcal{P}_\alpha] (-1)^{\varepsilon_\alpha}$ represent group generators within a BRST framework for Lie group theories [8] we expect $\rho(\phi) = \mathcal{P}_\alpha \phi^\alpha$. The above scenario is exactly true for theories with rank zero and one. However, for open groups of rank two and higher $\tilde{\theta}_\alpha$ satisfy a closed algebra only together with \mathcal{P}_α . This means that the final picture in this case must be somewhat different, which indeed is the case as we shall see. Let us now turn to the actual derivation.

Actual derivation

The starting point in our derivation is the Lie equations

$$\langle A(\phi) | \overleftarrow{D}_\alpha \equiv \langle A(\phi) | \left(\overleftarrow{\partial}_\alpha - (i\hbar)^{-1} Y_\alpha(\phi) \right) = 0,$$

$$A(\phi) \overleftarrow{\nabla}_a \equiv A(\phi) \overleftarrow{\partial}_a - (i\hbar)^{-1} [A(\phi), Y_a(\phi)] = 0, \quad (8)$$

where $Y_\alpha(\phi)$ is a connection operator. The integrability conditions for $Y_\alpha(\phi)$ are

$$Y_\alpha \overset{\leftarrow}{\partial}_\beta - Y_\beta \overset{\leftarrow}{\partial}_\alpha (-1)^{\varepsilon_\alpha \varepsilon_\beta} = (i\hbar)^{-1} [Y_\alpha, Y_\beta]. \quad (9)$$

The above scenario then suggests that

$$Y_\alpha(\phi) = (i\hbar)^{-1} [\Omega, \Omega_\alpha(\phi)], \quad (10)$$

where $\Omega_\alpha(0) = \mathcal{P}_\alpha$. The integrability conditions (9) for Y_α imply then

$$[\Omega, \Omega_\alpha \overset{\leftarrow}{\partial}_\beta - \Omega_\beta \overset{\leftarrow}{\partial}_\alpha (-1)^{\varepsilon_\alpha \varepsilon_\beta} - (i\hbar)^{-2} (\Omega_\alpha, \Omega_\beta)_\Omega] = 0, \quad (11)$$

where I have introduced

The quantum antibracket

The quantum antibracket is defined by [9, 10]

$$(f, g)_Q \equiv \frac{1}{2} \left([f, [Q, g]] - [g, [Q, f]] (-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} \right), \quad (12)$$

where Q is any odd operator. The quantum antibracket is a new general algebraic tool in terms of which one *e.g.* may give an operator version of the BV-quantization of general gauge theories [9, 10]. It generalizes the classical antibracket to operators. In fact, it reduces to the classical antibracket when f and g are functions and when Q is a second order nilpotent differential operator. The quantum antibracket satisfies the algebraic properties of the classical antibracket except that we have a generalized Leibniz' rule:

$$(fg, h)_Q - f(g, h)_Q - (f, h)_Q g (-1)^{\varepsilon_g(\varepsilon_h+1)} = \frac{1}{2} \left([f, h][g, Q] (-1)^{\varepsilon_h(\varepsilon_g+1)} + [f, Q][g, h] (-1)^{\varepsilon_g} \right). \quad (13)$$

The ordinary Leibniz' rule is only satisfied for commuting operators. The quantum antibracket also satisfies generalized Jacobi identities. For $Q^2 = 0$ we have

$$(f, (g, h)_Q) (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} + \text{cycle}(f, g, h) = -\frac{1}{2} [(f, g, h)_Q (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)}, Q], \quad (14)$$

where $(f, g, h)_Q$ is an example of higher quantum antibrackets defined by [10]

$$(f_{a_1}, \dots, f_{a_n})_Q \equiv -Q(\lambda) \overset{\leftarrow}{\partial}_{a_1} \overset{\leftarrow}{\partial}_{a_2} \cdots \overset{\leftarrow}{\partial}_{a_n} (-1)^{E_n} \Big|_{\lambda=0}, \quad (15)$$

where the derivatives, ∂_a , are with respect to the parameters, λ^a , and where

$$Q(\lambda) \equiv e^{-A} Q e^A, \quad A \equiv f_a \lambda^a, \quad E_n \equiv \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \varepsilon_{a_{2k+1}}, \quad (16)$$

where in turn $\varepsilon_a \equiv \varepsilon(f_a) = \varepsilon(\lambda^a)$.

Back to the integrability conditions (11)

The integrability conditions (11) for Ω_α may be cast into the form

$$\Omega_\alpha \overset{\leftarrow}{\partial}_\beta - \Omega_\beta \overset{\leftarrow}{\partial}_\alpha (-1)^{\varepsilon_\alpha \varepsilon_\beta} - (i\hbar)^{-2} (\Omega_\alpha, \Omega_\beta)_\Omega + \frac{1}{2} (i\hbar)^{-1} [\Omega_{\alpha\beta}, \Omega] = 0. \quad (17)$$

That there must be a BRST exact solution of (11) is due to the fact that Ω_α has negative ghost number. From these integrability conditions we may then derive integrability conditions for $\Omega_{\alpha\beta}$ given by

$$\begin{aligned} & \partial_\alpha \Omega_{\beta\gamma} (-1)^{\varepsilon_\alpha \varepsilon_\gamma} + \frac{1}{2} (i\hbar)^{-2} (\Omega_\alpha, \Omega_{\beta\gamma})_\Omega (-1)^{\varepsilon_\alpha \varepsilon_\gamma} + \text{cycle}(\alpha, \beta, \gamma) = \\ & = -(i\hbar)^{-3} (\Omega_\alpha, \Omega_\beta, \Omega_\gamma)_\Omega (-1)^{\varepsilon_\alpha \varepsilon_\gamma} - \frac{2}{3} (i\hbar)^{-1} [\Omega_{\alpha\beta\gamma}, \Omega]. \end{aligned} \quad (18)$$

We may then proceed and derive integrability conditions for $\Omega_{\alpha\beta\gamma}$ which introduces Ω 's with still more indices. Now Ω 's with higher indices are nonzero only for theories of rank two and higher which demonstrates that the starting scenario in section 3 then is incorrect. $\tilde{\theta}_\alpha$ do not close in this case.

The quantum master equation

The original simple integrability conditions (9) for the operator connections Y_α are through the basic assumption (10) replaced by an infinite set of integrability conditions for an infinite set of operators. One may question what we have gained. However, at this point a miracle happened. Batalin suggested that there must be a master equation for all these integrability conditions and indeed this is the case. This master equation is

$$(S, S)_\Delta = i\hbar [\Delta, S], \quad (19)$$

where

$$\Delta \equiv \Omega + \eta^\alpha \pi_\alpha (-1)^{\varepsilon_\alpha}, \quad \Delta^2 = 0, \quad (20)$$

is an extended BRST charge operator (π_α are conjugate momenta to ϕ^α), and where

$$\begin{aligned} S(\phi, \eta) & \equiv G + \eta^\alpha \Omega_\alpha(\phi) + \frac{1}{2} \eta^\beta \eta^\alpha \Omega_{\alpha\beta}(\phi) (-1)^{\varepsilon_\beta} + \\ & \frac{1}{3!} \eta^\gamma \eta^\beta \eta^\alpha \Omega_{\alpha\beta\gamma}(\phi) (-1)^{\varepsilon_\beta + \varepsilon_\alpha \varepsilon_\gamma} + \\ & \dots + \frac{1}{n!} \eta^{\alpha_n} \dots \eta^{\alpha_1} \Omega_{\alpha_1 \dots \alpha_n}(\phi) (-1)^{\varepsilon_n} + \dots \end{aligned} \quad (21)$$

is a master charge operator. η^α are new ghosts, which may be interpreted as ghosts or superpartners to the group parameters ϕ^α . The equivalence between the master equation (19) and the set of integrability conditions for the Ω 's has been checked up to third order in η^α . In fact, $\Omega_{\alpha\beta\gamma}$ in (18) and (21) are not the same. They are related as follows

$$\Omega'_{\alpha\beta\gamma} \equiv \Omega_{\alpha\beta\gamma} - \frac{1}{8} \left\{ (i\hbar)^{-1} [\Omega_{\alpha\beta}, \Omega_\gamma] (-1)^{\varepsilon_\alpha \varepsilon_\gamma} + \text{cycle}(\alpha, \beta, \gamma) \right\}, \quad (22)$$

where $\Omega'_{\alpha\beta\gamma}$ is equal to $\Omega_{\alpha\beta\gamma}$ in (18).

The quantum master equation (19) is quite different from the quantum master equation in the BV-quantization of general gauge theories which is

$$\frac{1}{2}(S, S) = i\hbar\Delta S, \quad (23)$$

where S is the master action which is a functional of fields and antifields, and where the Δ -operator is the following second order differential operator

$$\Delta \equiv (-1)^{\varepsilon_A} \frac{\overrightarrow{\partial}}{\partial \phi^A} \frac{\overrightarrow{\partial}}{\partial \phi_A^*}. \quad (24)$$

The bracket in (23) is the classical antibracket. In contrast the master equation (19) is entirely expressed in terms of operators. We do not know how the quantum master equations (19) and (23) are related if they are related at all. However, there is a formal similarity between the Δ -operators (20) and (24) for $\Omega = 0$ if one introduces conjugate momenta to the new ghosts η^α .

Interpretation of the quantum master equation

The new quantum master equation (19) encodes generalized Maurer-Cartan equations. In the case of Lie groups we have

$$S(\phi, \eta) \equiv G + \eta^\alpha \lambda_\alpha^\beta(\phi) \mathcal{P}_\beta, \quad (25)$$

which when inserted into (19) yields the standard Maurer-Cartan equations

$$\partial_\alpha \lambda_\beta^\gamma - \partial_\beta \lambda_\alpha^\gamma (-1)^{\varepsilon_\alpha \varepsilon_\beta} = \lambda_\alpha^\eta \lambda_\beta^\delta U_{\delta\eta}^\gamma (-1)^{\varepsilon_\beta \varepsilon_\eta + \varepsilon_\gamma + \varepsilon_\delta + \varepsilon_\eta}. \quad (26)$$

For quasi groups which satisfy the properties [6]

$$[U_{\alpha\beta}^\gamma, U_{\delta\epsilon}^f] = 0, \quad [[\theta_\delta, U_{\alpha\beta}^\gamma], U_{\epsilon\zeta}^\eta] = 0, \quad (27)$$

we still have (25) and (26). However, λ_β^α in (25) is then an operator and λ_β^α in (26) is replaced by a transformed operator (see [1]).

Solution of the quantum master equation

Define transformed master charges and Δ -operators by

$$S(\alpha) \equiv e^{\frac{i}{\hbar}\alpha F} S e^{-\frac{i}{\hbar}\alpha F}, \quad \Delta(\alpha) \equiv e^{\frac{i}{\hbar}\alpha F} \Delta e^{-\frac{i}{\hbar}\alpha F}, \quad (28)$$

where α is a parameter and F an operator. These transformed operators satisfy then the quantum master equation

$$(S(\alpha), S(\alpha))_{\Delta(\alpha)} = i\hbar[\Delta(\alpha), S(\alpha)]. \quad (29)$$

If we restrict the transformations such that $\Delta(\alpha) = \Delta$ then $S(\alpha)$ represents a different solution to the quantum master equation than S . Now $\Delta(\alpha) = \Delta$ requires $[\Delta, F] = 0$ whose solution is

$$F(\phi, \eta) = F(0, 0) + (i\hbar)^{-1}[\Delta, \Psi(\phi, \eta)]. \quad (30)$$

For $F(0, 0) = 0$ we have therefore the following natural invariance transformation

$$S \rightarrow S' \equiv \exp \left\{ -(i\hbar)^{-2} [\Delta, \Psi] \right\} S \exp \left\{ (i\hbar)^{-2} [\Delta, \Psi] \right\}. \quad (31)$$

Now since $S = G$ is a particular solution of the master equation (19) we conclude that the general solution must be

$$S = \exp \left\{ -(i\hbar)^{-2} [\Delta, \Psi] \right\} G \exp \left\{ (i\hbar)^{-2} [\Delta, \Psi] \right\}. \quad (32)$$

Notice that this S satisfies the boundary condition $S(0, 0) = G$ as required by the general form (21).

Explicit representation of open group transformations

The general solution (32) suggests that

$$U(\phi, \eta) \equiv \exp \left\{ -(i\hbar)^{-2} [\Delta, \Psi(\phi, \eta)] \right\} \quad (33)$$

should represent appropriate group elements for arbitrary open groups. Group transformed states and operators are then defined by ($U(0, 0) = \mathbf{1}$)

$$\langle \tilde{A}(\phi, \eta) | \equiv \langle A | U^{-1}(\phi, \eta), \quad \tilde{A}(\phi, \eta) \equiv U(\phi, \eta) A U^{-1}(\phi, \eta). \quad (34)$$

Notice that the master charge itself is a group transformed ghost charge, which implies

$$\begin{aligned} G|A\rangle_g = i\hbar g|A\rangle_g &\Rightarrow S|\tilde{A}\rangle_g = i\hbar g|\tilde{A}\rangle_g, \\ [G, A_g] = i\hbar g A_g &\Rightarrow [S, \tilde{A}_g] = i\hbar g \tilde{A}_g. \end{aligned} \quad (35)$$

The group transformed states and operators (34) satisfy the extended Lie equations

$$\begin{aligned} \langle \tilde{A}(\phi, \eta) | \overset{\leftarrow}{D}_\alpha &\equiv \langle \tilde{A}(\phi, \eta) | \left(\overset{\leftarrow}{\partial}_\alpha - (i\hbar)^{-1} \tilde{Y}_\alpha(\phi, \eta) \right) = 0, \\ \tilde{A}(\phi, \eta) \overset{\leftarrow}{\nabla}_\alpha &\equiv \tilde{A}(\phi, \eta) \overset{\leftarrow}{\partial}_\alpha - (i\hbar)^{-1} [\tilde{A}(\phi, \eta), \tilde{Y}_\alpha(\phi, \eta)] = 0, \end{aligned} \quad (36)$$

where

$$\tilde{Y}_\alpha(\phi, \eta) = (i\hbar)^{-1} [\Delta, \tilde{\Omega}_\alpha(\phi, \eta)] = i\hbar U(\phi, \eta) \left(U^{-1}(\phi, \eta) \overset{\leftarrow}{\partial}_\alpha \right). \quad (37)$$

A natural explicit form of (33) is obtained for the choice $\Psi(\phi, \eta) = \phi^\alpha \mathcal{P}_\alpha$ which yields

$$U(\phi, \eta) = \exp \left\{ \frac{i}{\hbar} (\phi^\alpha \tilde{\theta}_\alpha - \eta^\alpha \mathcal{P}_\alpha) \right\}. \quad (38)$$

Notice that $\tilde{\theta}_\alpha$ and \mathcal{P}_α satisfy a closed algebra. For theories with rank zero and one we may set $\eta^\alpha = 0$ in which case we obtain the group element we started from in section 3.

An interpretation

In a BRST quantization of general gauge theories we notice the properties

$$\begin{aligned}\Omega|A\rangle = 0 &\Rightarrow \Delta|\tilde{A}(\phi, \eta)\rangle = 0, \\ [\Omega, A] = 0 &\Rightarrow [\Delta, \tilde{A}(\phi, \eta)] = 0.\end{aligned}\tag{39}$$

Since $|\tilde{A}(0, 0)\rangle = |A\rangle$, $\tilde{A}(0, 0) = A$, this may be interpreted as if we have a BRST theory with Δ as the BRST charge. The extended BRST charge Δ looks like a BFV-BRST charge in the nonminimal sector where ϕ^α are Lagrange multipliers and η^α antighost momenta, *i.e.* Δ is the appropriate BRST charge for path integrals in the Hamiltonian BFV formulation.

The $\text{Sp}(2)$ formulation [4]

Arbitrary involutions

$$\{\theta_\alpha(z), \theta_\beta(z)\} = U_{\alpha\beta}{}^\gamma(z)\theta_\gamma(z)\tag{40}$$

may also be embedded into a BRST and antiBRST charge. In the $\text{Sp}(2)$ -version [11] these charges are denoted, Ω^a , $a = 1, 2$, and satisfy at the quantum level

$$[\Omega^a, \Omega^b] = \Omega^{\{a}\Omega^{b\}} = 0.\tag{41}$$

Open group transformations may also be derived within this framework. In fact, all previous results may be extended to this case [4]. First we imagine that finite open group transformations are represented by group elements of the form

$$U(\phi) = \exp\{-(i\hbar)^{-2}\varepsilon_{ab}[\Omega^b, [\Omega^a, R(\phi)]]\},\tag{42}$$

where ε_{ab} is the $\text{Sp}(2)$ metric ($\varepsilon_{21} = -\varepsilon_{12} = 1$). Then we assume that group transformed states and operators satisfy the Lie equations (8). Due to (42) the operator connections $Y_\alpha(\phi)$ should therefore have the form

$$Y_\alpha = (i\hbar)^{-2}\frac{1}{2}\varepsilon_{ab}[\Omega^b, [\Omega^a, X_\alpha(\phi)]].\tag{43}$$

In order for Y_α to satisfy the same boundary condition as before, *i.e.* (10), the $\text{Sp}(2)$ charge Ω^a must be given in the non-minimal sector with dynamical Lagrange multipliers. Explicitly this means [11]

$$\Omega^a = \mathcal{C}^{\alpha a}\theta_\alpha + \frac{1}{2}\mathcal{C}^{\beta b}\mathcal{C}^{\alpha a}U_{\alpha\beta}{}^\gamma\mathcal{P}_{\gamma b}(-1)^{\varepsilon_\beta+\varepsilon_\gamma} + \varepsilon^{ab}\mathcal{P}_{\beta b}\lambda^\beta + \frac{1}{2}\lambda^\beta\mathcal{C}^{\alpha a}U_{\alpha\beta}{}^\gamma\zeta_\gamma + \dots,\tag{44}$$

where $\mathcal{C}^{\alpha a}$ and $\mathcal{P}_{\alpha a}$ are $\text{Sp}(2)$ ghosts and their conjugate momenta, and where λ^α and ζ_α are the Lagrange multipliers and their conjugate momenta. Their commutation relations are

$$[\mathcal{C}^{\alpha a}, \mathcal{P}_{\beta b}] = i\hbar\delta_\beta^\alpha\delta_b^a, \quad [\lambda^\alpha, \zeta_\beta] = i\hbar\delta_\beta^\alpha.\tag{45}$$

The expression (44) represents the first terms in a \mathcal{CP} - and $\lambda\zeta$ -ordered $\text{Sp}(2)$ charge. The remaining terms are determined by the conditions (41). In terms of these charges the

connection operators Y_α are realized according to the formula (43) with the boundary condition $X_\alpha(0) = -\zeta_\alpha$. The integrability conditions (9) for Y_α imply here

$$X_\alpha \overleftarrow{\partial}_\beta - X_\beta \overleftarrow{\partial}_\alpha (-1)^{\varepsilon_\alpha \varepsilon_\beta} + (i\hbar)^{-3} \frac{1}{2} \{X_\alpha, X_\beta\}_\Omega = (i\hbar)^{-1} [X_{\alpha\beta a}, \Omega^a], \quad (46)$$

where

$$\{f, g\}_Q \equiv \varepsilon_{ab} [[f, Q^a], [Q^b, g]] \quad (47)$$

is a generalized commutator. From (46) one may derive integrability conditions for $X_{\alpha\beta a}$ which introduces X 's with higher indices etc. However, also in this case a miracle happens.

The $\text{Sp}(2)$ quantum master equation

The integrability conditions for X_α , $X_{\alpha\beta a}$ etc may be embedded into the quantum master equation

$$(S, S)_\Delta^a = i\hbar [\Delta^a, S], \quad (48)$$

where

$$(f, g)_\Delta^a \equiv \frac{1}{2} \left([f, [\Delta^a, g]] - [g, [\Delta^a, f]] (-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} \right) \quad (49)$$

is the $\text{Sp}(2)$ quantum antibracket which generalizes the classical $\text{Sp}(2)$ antibracket [12], and where

$$\Delta^a \equiv \Omega^a + \eta^{\alpha a} \pi_\alpha (-1)^{\varepsilon_\alpha} + \rho^\alpha \xi_{\alpha b} \varepsilon^{ab} (-1)^{\varepsilon_\alpha} \quad (50)$$

is an extended $\text{Sp}(2)$ charge which satisfies

$$[\Delta^a, \Delta^b] = \Delta^{\{a} \Delta^{b\}} = 0. \quad (51)$$

The master charge S in (48) is here given by

$$\begin{aligned} S(\phi, \rho, \eta) = & G + \eta^{\alpha a} \Omega_{\alpha a}(\phi) + \rho^\alpha \Omega_\alpha(\phi) + \frac{1}{2} \eta^{\beta b} \eta^{\alpha a} (-1)^{\varepsilon_\beta} \Omega_{\alpha\beta ab}(\phi) + \\ & + \frac{1}{2} \rho^\beta \rho^\alpha \Omega_{\alpha\beta}(\phi) + \rho^\beta \eta^{\alpha a} \Omega_{\alpha\beta a}(\phi) + \dots, \end{aligned} \quad (52)$$

where ϕ^α, ρ^α , and $\eta^{\alpha a}$ constitute a supersymmetric set of variables. Their Grassmann parities are $\varepsilon(\phi^\alpha) = \varepsilon(\rho^\alpha) = \varepsilon_\alpha$ and $\varepsilon(\eta^{\alpha a}) = \varepsilon_\alpha + 1$. $\xi_{\alpha b}$ in (50) are conjugate momenta to $\eta^{\alpha a}$.

The master charge (52) is given by a general power expansion in ρ^α and $\eta^{\alpha a}$. The coefficient operators have the following identifications with the X 's in the integrability conditions

$$\begin{aligned} X_\alpha &= \frac{1}{2} \Omega_\alpha (-1)^{\varepsilon_\alpha}, \\ X_{\alpha\beta a} &\equiv \frac{1}{6} \left(\Omega_{\alpha\beta a} (-1)^{\varepsilon_\alpha} - \Omega_{\beta\alpha a} (-1)^{\varepsilon_\beta(\varepsilon_\alpha+1)} + (i\hbar)^{-2} \varepsilon_{ab} [[X_\alpha, X_\beta], \Omega^b] \right). \end{aligned} \quad (53)$$

The master equation yields furthermore

$$\Omega_{\alpha a} = \frac{1}{2}(i\hbar)^{-1}\varepsilon_{ab}[\Omega_\alpha, \Omega^b] \quad (54)$$

and determines $\Omega_{\alpha\beta ab}$ in terms of $\Omega^a, \Omega_{\alpha a}$ and $\Omega_{\alpha\beta a}$.

Even here there is a similar looking quantum master equation to (48) within the $\text{Sp}(2)$ extended BV quantization [12]. However, it is given for the master action S and in terms of classical antibrackets. Δ^a are second order differential operators satisfying (51) with only a formal similarity to (50) with $\Omega^a = 0$.

Solution of the $\text{Sp}(2)$ quantum master equation

Define as before transformed master charges and Δ operators according to

$$S(\alpha) \equiv e^{\frac{i}{\hbar}\alpha F} S e^{-\frac{i}{\hbar}\alpha F}, \quad \Delta^a(\alpha) \equiv e^{\frac{i}{\hbar}\alpha F} \Delta^a e^{-\frac{i}{\hbar}\alpha F}. \quad (55)$$

They obviously satisfy the master equation

$$(S(\alpha), S(\alpha))_{\Delta(\alpha)}^a = i\hbar[\Delta^a(\alpha), S(\alpha)]. \quad (56)$$

In order to derive general solutions of (48) we impose the restriction $\Delta^a(\alpha) = \Delta^a$ which requires $[\Delta^a, F] = 0$. The solution is

$$F(\phi, \eta, \rho) = F(0, 0, 0) + \frac{1}{2}\varepsilon_{ab}(i\hbar)^{-2}[\Delta^b, [\Delta^a, \Phi(\phi, \eta, \rho)]]. \quad (57)$$

For $F(0, 0, 0) = 0$ we have the natural invariance transformation

$$S \rightarrow S' \equiv \exp\left\{-(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]]\right\} S \exp\left\{(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]]\right\}. \quad (58)$$

Since $S = G$ is a particular solution of the master equation, we have therefore the general solution

$$S = \exp\left\{-(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]]\right\} G \exp\left\{(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[\Delta^b, [\Delta^a, \Phi]]\right\}. \quad (59)$$

Explicit representation of open group transformations within the $\text{Sp}(2)$ scheme

The result (59) suggests that group elements within the $\text{Sp}(2)$ scheme should be defined as follows ($U(0, 0, 0) = \mathbf{1}$)

$$U(\phi, \eta, \rho) \equiv \exp\left\{-\frac{1}{2}\varepsilon_{ab}(i\hbar)^{-3}[\Delta^b, [\Delta^a, \Phi(\phi, \eta, \rho)]]\right\}. \quad (60)$$

Transformed states and operators are then given by

$$\langle \tilde{A}(\phi, \eta, \rho) | \equiv \langle A | U^{-1}(\phi, \eta, \rho), \quad \tilde{A}(\phi, \eta, \rho) \equiv U(\phi, \eta, \rho) A U^{-1}(\phi, \eta, \rho). \quad (61)$$

It follows then that the master charge S is a group transformed ghost charge. The properties (35) are therefore valid also here.

A natural explicit form of $U(\phi, \eta, \rho)$ is obtained from (60) with the choice $\Phi(\phi, \eta, \rho) = \phi^\alpha \zeta_\alpha$. It is

$$U(\phi, \eta) = \exp \left\{ \frac{i}{\hbar} (\phi^\alpha \tilde{\theta}_\alpha + \eta^{\alpha\alpha} \tilde{\mathcal{P}}_{\alpha\alpha} - \rho^\alpha \zeta_\alpha) \right\}, \quad (62)$$

where

$$\tilde{\theta}_\alpha \equiv (i\hbar)^{-2} \frac{1}{2} \varepsilon_{ab} [\Omega^b, [\Omega^a, \zeta_\alpha]], \quad \tilde{\mathcal{P}}_{\alpha\alpha} \equiv (i\hbar)^{-1} \varepsilon_{ab} [\Omega^b, \zeta_\alpha] (-1)^{\varepsilon_\alpha}. \quad (63)$$

ζ_α , $\tilde{\theta}_\alpha$ and $\tilde{\mathcal{P}}_{\alpha\alpha}$ should satisfy a closed algebra.

Conclusions

I hope I have been able to convey the message that it seems possible to develop a group theory even for open groups. I expect the results obtained so far only to constitute a beginning of such a development. Many consequences remain to be drawn. There are probably also several reinterpretations of known properties awaiting to be discovered.

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